SURGERY OF COMPLEX ANALYTIC DYNAMICAL SYSTEMS

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SURGERY OF COMPLEX ANALYTIC DYNAMICAL SYSTEMS

Mitsuhiro Shishikura (宍倉光広)

Department of Mathematics
Faculty of Science
Kyoto University
Kyoto, 606, Japan

ABSTRACT A method of quasi-conformal surgery of rational functions is proposed. By the surgery, rational functions with Herman rings are constructed. Such a function is found by a numerical experiment. The method is also applied to solving the problem of the number of stable regions.

§1. INTRODUCTION

Let us consider a complex analytic dynamical system $f:\overline{\mathbf{C}} \to \overline{\mathbf{C}}$, where f is a rational function with degree ≥ 2, and $\overline{\mathbf{C}}$ is the Riemann sphere. The *stable set* of f is the maximal open set where the iterations f^n $(n \geq 0)$ are equicontinuous. Each connected component of the stable set is called the *stable region*. D.Sullivan has proved that each stable region is eventually cyclic by f, and that the cyclic stable regions are classified into the follwing types: attractive basin, parabolic basin, Siegel disk and Herman ring. (See [3], [8] and also §3.)

In this paper, we deal with a method of surgery to construct some new rational functions from given functions, using quasi-conformal mappings(see §2). We call this *quasi-conformal surgery* (or qc-surgery). Such technique was first introduced by A.Douady and J.H.Hubbard [4].

Although their surgery concerned only with attractive basins, the idea is applicable in various other situations. We shall formulate the qc-surgery more generally, and apply it to some existence problems of rational functions, especially to the construction of Herman rings:

THEOREM 1. For all integer $p \geqq 1$, there exists a rational function of degree 3, which has Herman rings of order p.

To prove this theorem, we cut some rational functions with Siegel disks along their invariant curves, and glue them up (see §4). M.R.Herman[5] proved the same result without stating its degree, by a different method (see §3). Our method shows immediately:

THEOREM 2. For all irrational number θ, there exists a rational function with a Siegel disk of rotation number θ, if and only if there exists a rational function with a Herman ring of rotation number θ.

We seek a concrete example of the function in theorem 1 (p=2) by a numerical experiment(§5).

As another application of the qc-surgery, we have :

THEOREM 3. A rational function of degree d has at most $2(d-1)$ cycles of stable regions, with each Herman ring counted twice. Moreover, there exist at most $(d-2)$ Herman rings.

Conversely, given the number of cycles for each type of stable regions, satisfying the above conditions, one can find a rational function of degree d, which has those numbers of cycles.

This theorem answers to a problem in [8] (or problem 7.8 in [3]).

And it follows that a rational function of degree 2 has no Herman ring. A sketch of proof of the theorem is given in §6. For complete proof, see [7].

§2. FUNDAMENTAL LEMMA FOR QC-SURGERY

Definitions. Let Ω, Ω' be domains of **C**. A homeomorphism $\phi:\Omega \to \Omega'$ is a *quasi-conformal mapping* (qc-mapping) if ϕ is absolutely continuous on almost all lines parallel to real-axis and on almost all lines parallel to imaginary-axis, and $|\phi_{\bar{z}}/\phi_z| \leq k$ a.e.(almost everywhere w.r.t. Lebesgue measure), for some $k < 1$. (cf. Ahlfors[1].)

A mapping $f:\Omega \to \overline{\mathbf{C}}$ is *quasi-regular*, if it is a composition of a qc-mapping and an analytic mapping.

Quasi-conformal mappings (or quasi-regular mapping) on Riemann surfaces are defined similarly by means of local charts.

Here is the formulation of the qc-surgery :

LEMMA.(Fundamental lemma for qc-surgery)

Let $g:\overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be a quasi-regular mapping. Suppose that there are disjoint open sets E_i of $\overline{\mathbf{C}}$, qc-mappings $\Phi_i:E_i \to E_i' \subset \overline{\mathbf{C}}$ $(i=1,\dots,m)$ and integer $N \geq 0$, satisfying following conditions :

(i) $g(E) \subset E$, where $E = E_1 \cup \dots \cup E_m$;

(ii) $\Phi \circ g \circ \Phi_i^{-1}$ is analytic on E_i', where we define $\Phi:E \to \overline{\mathbf{C}}$ by $\Phi|_{E_i} = \Phi_i$;

(iii) $g_{\bar{z}} = 0$ a.e. on $\overline{\mathbf{C}}-g^{-N}(E)$.

Then there exists a qc-mapping ϕ of $\overline{\mathbf{C}}$ such that $\phi \circ g \circ \phi^{-1}$ is a rational function.

Proof.(see [4], [8]) Define a measurable conformal structure σ on $\overline{\mathbf{C}}$ as follows. Let σ_0 be the conformal structure defined by $|dz|$. We set $\sigma = \Phi^*\sigma_0$ on E, where $\Phi^*\sigma_0$ means the pull-back of σ_0 by Φ, defined except on a null set. The invariance (a.e.) of σ by g follows from (ii). On $\bigcup_{n=0}^{\infty} g^{-n}(E)$, define σ by pulling back $\sigma|_E$ by g. Finally, set $\sigma = \sigma_0$ on the remaining part of $\overline{\mathbf{C}}$.

Thus we have defined the σ a.e. on $\overline{\mathbf{C}}$, which is a.e. invariant by g from the definition and (iii). Moreover, it follows from (iii) that the distortion of σ with respect to σ_0 is uniformly bounded. Hence, by the measurable mapping theorem (cf. [1]), there exists a quasi-conformal mapping ϕ of $\overline{\mathbf{C}}$ such that $\phi^*\sigma_0 = \sigma$ a.e. Then $f = \phi\circ g\circ\phi^{-1}$ preserves the standard conformal structure σ_0, a.e. So f is analytic on $\overline{\mathbf{C}}$, therefore, a rational function.

§3. SIEGEL DISKS AND HERMAN RINGS

Definition. A stable region D of a rational function f is a *Siegel disk* (resp. a *Herman ring*), if $f^p(D) = D$ for some $p \geq 1$ (the minimun of such p is called the *order* of D), and if there exist a conformal mapping $h: D \to \Delta = \{\zeta \in \mathbf{C}: |\zeta| < 1\}$ (resp. $h: D \to A_r = \{\zeta \in \mathbf{C}: r < |\zeta| < 1\}$, $0 < r < 1$) and an irrational number θ such that :

$$h\circ f^p(z) = e^{2\pi i\theta}\cdot h(z) \qquad \text{for } z \in D.$$

We call this θ the *rotation number* of D, which is determined modulo Z (and up to sign, in case of Herman ring). For a Siegel disk D, $h^{-1}(0)$ is called the *center* of D. One easily checks that for the center z_0,

$$f^p(z_0) = z_0 \quad \text{and} \quad (f^p)'(z_0) = e^{2\pi i\theta}. \tag{1}$$

Let us consider the converse. Suppose that a periodic point z_0 of a rational function f satisfies (1). If θ is Diophantine, i.e. if there exist constants $C, \varepsilon > 0$ such that

$$|\theta - p/q| \geq C/q^{2+\varepsilon} \qquad (p,q \in \mathbf{Z},\ q \geq 1),$$

then z_0 becomes the center of a Siegel disk of f. This was proved by C.L.Siegel[6], by solving a functional equation related to the linearization of f at z_0. On the other hand, H.Cremer proved that if θ is "sufficiently Liouville", z_0 cannot be the center of a Siegel disk. (see [3])

Concerning Herman ring, its existence was proved by M.R.Herman, for example, for a rational function of the form

$$z \to \frac{e^{i\alpha}}{z}\left(\frac{z-r}{1-rz}\right)^2, \qquad (2)$$

where $\alpha \in \mathbf{R}$ and $r>0$ small. He proved, furthermore, the existence of Herman rings of order $p > 1$, by his usual method against small denominators (see appendix of [5]). Compare it with our method carried out in §4.

§4. CONSTRUCTION OF HERMAN RINGS

Let $f_0, f_1, \ldots, f_p$ $(p \geq 1)$ be rational functions satisfying (see figure 1):

(a) f_0 has Siegel disks $S_1, \ldots, S_p$ of order p with rotation number θ, where $f_0(S_i) = S_{i+1}$ $(i=1,\ldots,p-1)$ and $f_0(S_p) = S_1$;

(b) the composite $f_p \circ \ldots \circ f_1$ has a Siegel disk S_1' of order 1 with rotation number $-\theta$.

Choose a (real analytic) Jordan curve γ_1 in S_1 invariant by

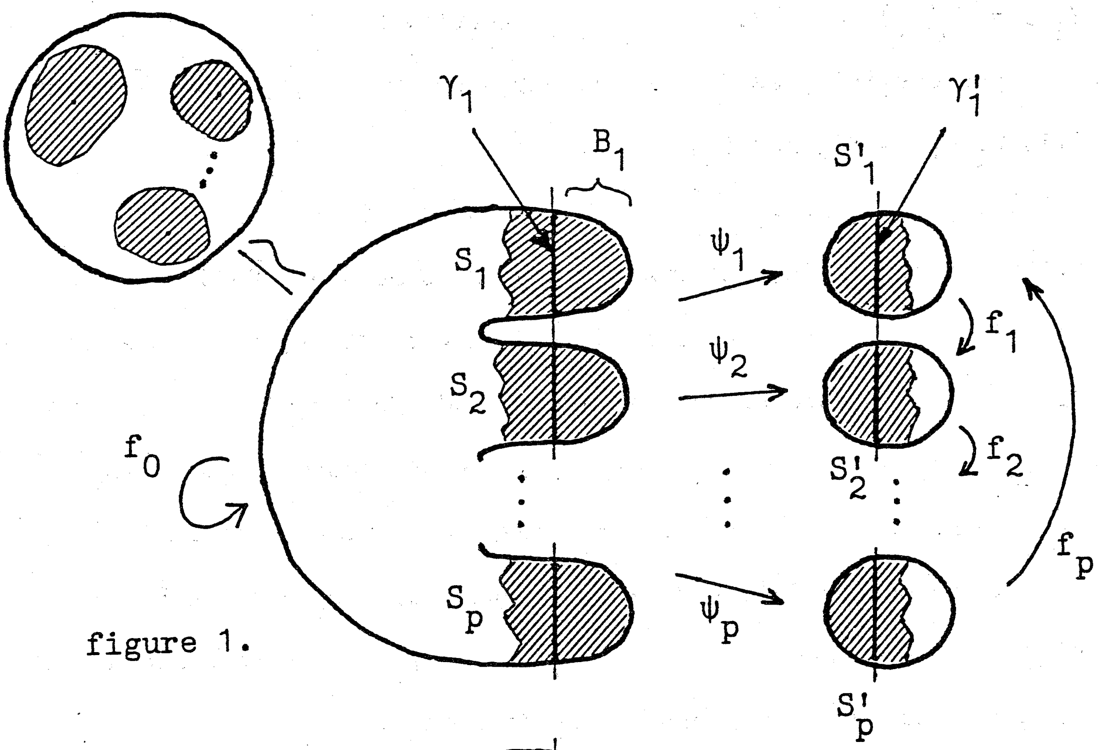


figure 1.

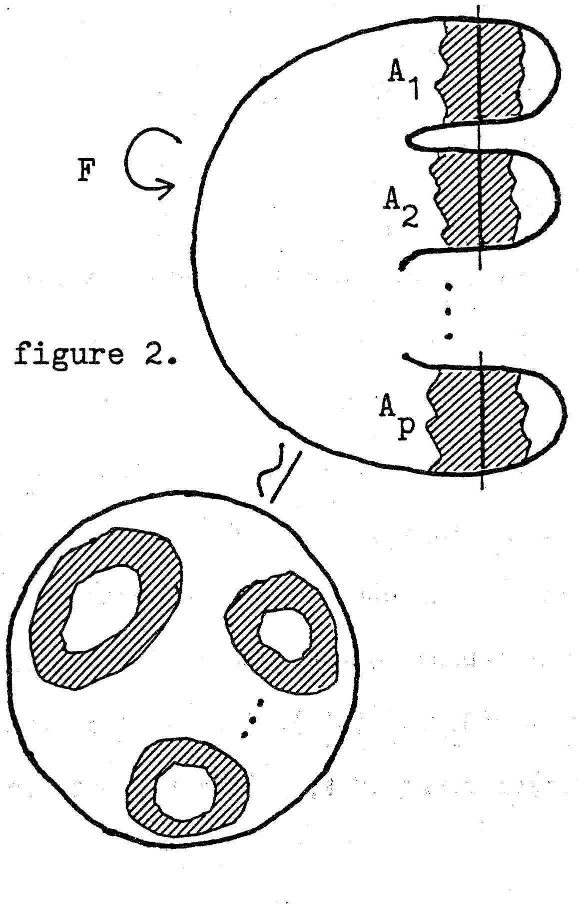
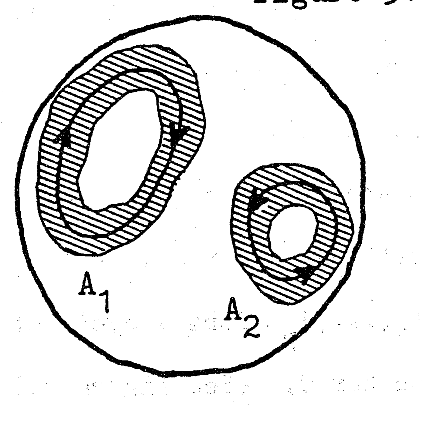


figure 2.

figure 3.



f_0, and γ_1' in S_1' invariant by $f_p \circ \cdots \circ f_1$. Define $S_{i+1}' = f_i(S_i')$, $\gamma_{i+1} = f_0(\gamma_i)$ and $\gamma_{i+1}' = f_i(\gamma_i')$, inductively.

There exist quasi-conformal mappings $\psi_1, \ldots, \psi_p : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ such that:

(i) $\psi_i(\gamma_i) = \gamma_i'$, for $i=1,\ldots,p$;

(ii) $\psi_{i+1} \circ f_0 = f_i \circ \psi_i$, on γ_i for $i=1,\ldots,p$, where $\psi_{p+1} = \psi_1$;

(iii) Each ψ_i is conformal in a neighborhood of $\overline{\mathbf{C}} - (S_i \cap \psi_i^{-1}(S_i'))$.

These ψ_i are constructed as follows: First, by the definition of Siegel disks, there exist real analytic diffeomorphisms $\psi_i : \gamma_i \to \gamma_i'$ satisfying (ii). Let B_i (resp. B_i') be the component of $\overline{\mathbf{C}} - \gamma_i$ (resp. $\overline{\mathbf{C}} - \gamma_i'$), entirely contained in S_i (resp. S_i'). Take conformal mappings (on each component) $\psi_i' : \overline{\mathbf{C}} - \gamma_i \to \overline{\mathbf{C}} - \gamma_i'$ such that $\psi_i'(B_i) = \overline{\mathbf{C}} - \overline{B_i'}$ and $\psi_i'(\overline{\mathbf{C}} - \overline{B_i}) = B_i'$. Modifying each ψ_i' near γ_i so as to coincide with the previous $\psi_i|_{\gamma_i}$, we obtain the desired ψ_i.

Now, define a mapping $g : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ by

$$g = \begin{cases} f_0 & \text{on } \overline{\mathbf{C}} - \bigcup_i B_i \\ \psi_{i+1}^{-1} \circ f_i \circ \psi_i & \text{on } B_i. \end{cases}$$

It is easily seen that g is continuous, and moreover, quasi-regular. Let $E_0 = \bigcup_{i=1}^{p} (S_i - \overline{B_i})$, $\Phi_0 = \mathrm{id}_{E_0}$,

$E_i = B_i \cap \psi_i^{-1}(S_i')$, $\Phi_i = \psi_i|_{E_i}$ $(i=1,\ldots,p)$,

and $N = 1$. Obviously, $g(E) = E$, where $E = \bigcup_i E_i$. As each ψ_i is conformal in a neighborhood of $\overline{\mathbf{C}} - (E \cup \gamma_i)$, $g_{\bar z} = 0$ a.e. in $\overline{\mathbf{C}} - g(E)$. Hence all the conditions in the fundamental lemma are satisfied. So there exists a quasi-conformal mapping ϕ such that $F = \phi \circ g \circ \phi^{-1}$ is a rational function. Write $A_i = \phi(S_i \cap \psi_i^{-1}(S_i'))$. It is clear that $A_1, \ldots, A_p$ form a cycle of Herman rings of F, of order p with rotation number θ. (See figure 2.)

Remark 1. A slightly different application of the lemma gives another rational function with Herman rings A_1, A_2 of order 2, as indicated in figure 3. Pay attention to the orientations of the invariant curves in A_i, and note that their disposition is different from the previous one for $p = 2$.

Remark 2. It is also possible to construct a rational function with a Siegel disk, from a given rational function with a Herman ring of order 1. In fact, we have only to put $p = 1$, $f_1(z) = e^{2\pi i\theta}\cdot z$ in the above.

In the case of order $\geqq 2$, such kind of surgery needs certain considerations on the disposition of Herman rings and their inverse images([7]).

Example. Let $f_0(z) = e^{2\pi i\theta}(z-1)^2/z$ and $f_1 = \tau\circ f_1\circ\tau$, where $\tau(z) = 1/\bar{z}$. If θ is Diophantine, f_0 (resp. f_1) has a Siegel disk of order 1 centered at ∞ (resp. 0). (see §3.) If the surgery is carried out "symmetically" w.r.t. $|z| = 1$, we can get a rational function of the form (2).

Proof of theorem 1. For this purpose, we take f_i so that $\deg f_0 = \deg f_1 = 2$, $f_2 = \ldots = f_p = id_{\bar{\mathbb{C}}}$, and of course, they satisfy (a) and (b) above. (Such f_i exist. see §3.) One can see that the obtained rational function F is of degree 3, by considering its topological degree.

Proof of theorem 2. If f has a Siegel disk of order p with rotation number θ, $f_0 = f^p$ (resp. $f_1(z) = \overline{f_0(\bar{z})}$) has a Siegel disk of order 1 with rotation number θ (resp. $-\theta$). Applying the above surgery to f_0 and f_1, we obtain F, which has a Herman ring of rotation number θ. The converse is proved similarly. See remark 2.

§5. A NUMERICAL EXPERIMENT ON HERMAN RINGS OF ORDER TWO

Herman rings of order 1 can be found for (2) rather easily, so we shall try to find a rational function with Herman rings of order 2, by a numerical experiment.

The construction in §4 not only assures us the existence of Herman rings of order p, but also suggests that if the γ_1 is chosen sufficiently close to the center of S_1, then the resulting rational function F will be close to f_0 outside a neighborhood of the centers (by suitable choice of ϕ). Hence, if $f_0(z) = z^2+c_0$ has a Siegel disk of order 2 with center z_0 (such c_0 can be found by an elementary calculation), it is expected that a rational function of the form

$$F(z) = F_{a,b,c}(z) = z^2 \cdot \frac{z-a}{z-b} + c$$

has a Herman rings of order 2, for suitable $a,b \fallingdotseq z_0$ and $c \fallingdotseq c_0$. Notice that F is close to f_0 outside a neiborhood of 0, if $a,b \fallingdotseq z_0$ and $c \fallingdotseq c_0$.

Here is an example of such parameters:

$c_0 = -0.8639244 +0.2103677\times i$, $z_0 = -0.0789079 -0.2497882\times i$,
$a = -0.0768679 -0.2503722\times i$, $b = -0.0809479 -0.2492042\times i$,
$c = -0.8648749 +0.2103377\times i$.

Fig.4 and fig.5 are the plots of orbits of critical points of f_0

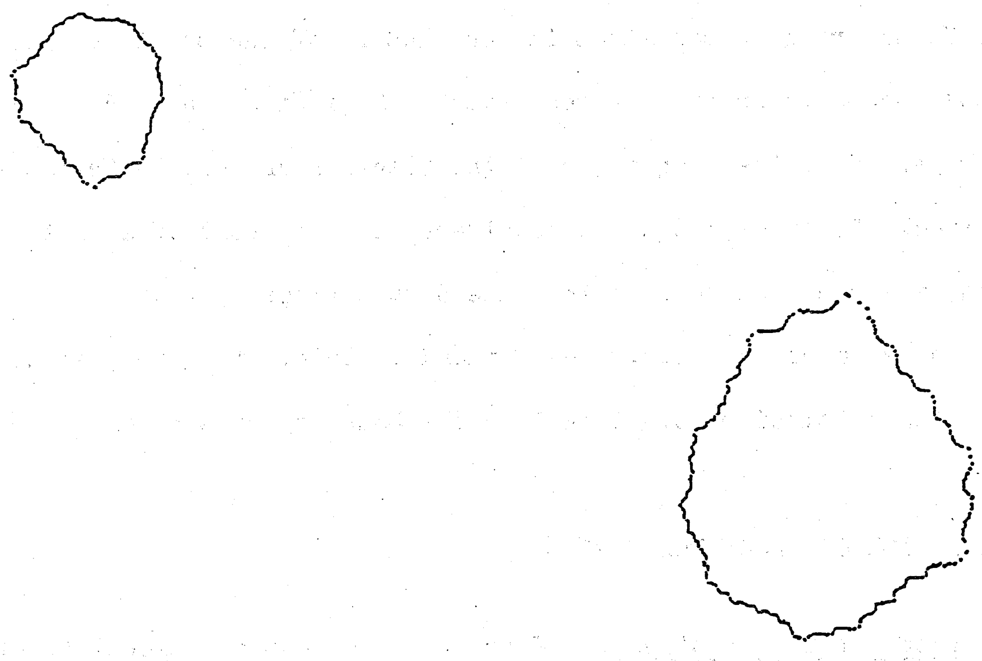


figure 4. Orbit of critical point 0 of f_0.

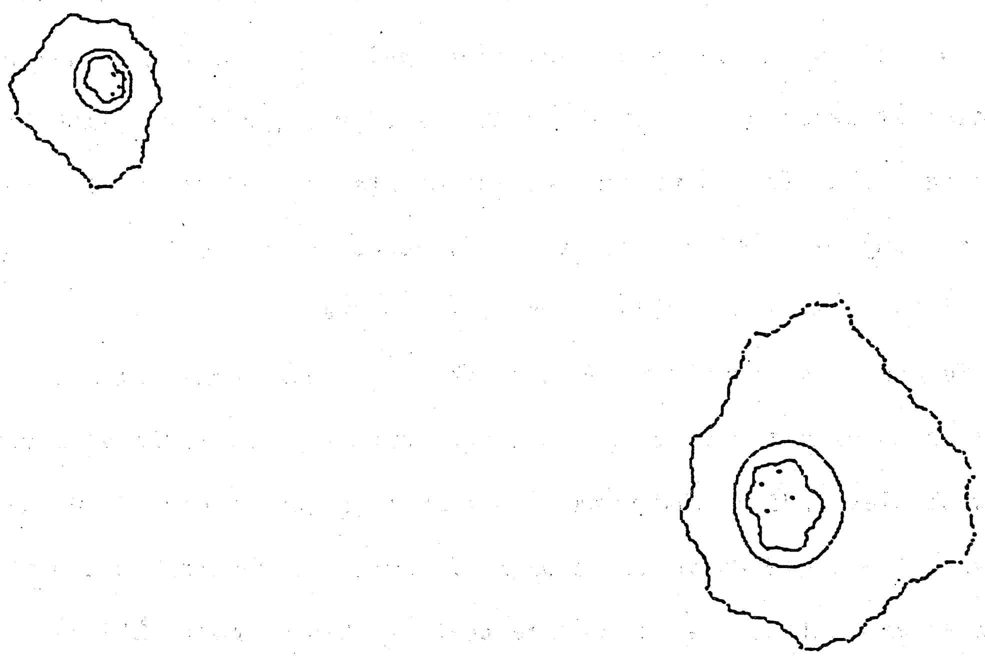


figure 5. Orbits of critical points of F.

and F, respectively. As it is known that the boundary of a Siegel disk or a Herman ring is contained in the closure of the orbits of critical points, the dots in the figures, except the middle "curves" in fig.5, are regarded as the boundaries of the Siegel disks or the Herman rings. The middle "curves" are to be considered as the orbit of a critical point, which happen to fall into the Herman rings.

These values of a, b, c were found by trial-and-error, so that the orbits of critical points look like the boundary of annular regions.

§6. SKETCH OF PROOF FOR THEOREM 3

Proof of the first half. First, assume that a rational function f of degree d has no Herman ring. One can perturb f so that all the indifferent (i.e. neither attractive nor repulsive) periodic points of f become attractive. (See the example below. There, we use the qc-surgery again.) It is known that each cycles of attractive basins contains at least one critical point. Its proof, combined with the above perturbation, shows that cycles of stable regions of f exist at most as many as critical points of f, hence at most $2(d-1)$. (Recall that f has at most $2(d-1)$ critical points.)

Secondly, we consider the case where f has Herman rings. Cutting f along invariant curves in the rings, one get some rational functions f_i with Siegel disks and with no Herman ring (see remark 2 in §4). Applying the above estimate to each of them, and summing up the thus obtained estimates, one can deduce that f has at most $2(d-1)$ cycles of stable regions. Notice that each Herman ring of f is counted twice as Siegel disks of the f_i.

Moreover if f has neither attractive basin, parabolic basin nor Siegel disk, the equality does not hold in the estimate. Therefore f has at most $d-2$ Herman rings.

Example of the perturbation. Suppose that f has 0 as an irrationally indifferent fixed point, i.e. $f(0) = 0$ and $f'(0) = e^{2\pi i\theta}$, where θ is irrational. Changing the coordinate, we may assume that $f(\infty) = 0$. Let $h(z)$ be a polynomial of degree k, satisfying $h(0) = 0$, $h'(0) = -1$. It follows from the theory of normal forms [2], that there exists an analytic local diffeomorphism ψ such that

$$\psi^{-1}\circ f\circ\psi(z) = e^{2\pi i\theta}\cdot z + O(z^{k+3}).$$

For small $\varepsilon > 0$, one can construct a quasi-conformal mapping H_ε of $\overline{\mathbf{C}}$, such that

$$H_\varepsilon(z) = z + \varepsilon h(z), \quad \text{for } |z| < (1/\varepsilon)^{1/k}.$$

Set $g_\varepsilon = f\circ H_\varepsilon$, $E = \psi(\{z: |z| < (1/\varepsilon)^{1/(k+1)}\})$, $\Phi = id_E$ and $N = 1$. It is easy to verify the conditions of the fundamental lemma for small $\varepsilon>0$, and the lemma yields a rational function $f_\varepsilon = \phi_\varepsilon\circ g_\varepsilon\circ\phi_\varepsilon^{-1}$, where ϕ_ε is a quasi-conformal mapping. If $\phi_\varepsilon(0) = 0$, 0 is an attractive fixed point of f_ε. We can choose h so that g_ε has the indifferent periodic points of f as attractive periodic points. Then f_ε are the desired perturbations of f.

Proof of the second half. Let

$$f(z) = z\cdot\frac{\lambda_1+z^{d-1}}{1+\lambda_2 z^{d-1}} \tag{3}$$

and p be an integer mutually prime with $d-1$. For certain $\lambda_1 \doteqdot 1$ and $\lambda_2 \doteqdot e^{2\pi i/p}$, f has $2(d-1)$ cycles of Siegel disks. The perturbation technique used in the proof of the first half, together with the surgery in remark 2 in § 4, enables us to construct, from rational functions of the form (3), the desired rational function.

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NORMAL FORMS FOR CONSTRAINED SYSTEMS

京大数学 岡 宏枝 (Hiroe OKA)

ABSTRACT

A global formulation with a coordinate-free description is given to ordinary differential equations (Abbrev. ODEs) including a small paramater ε. They are formulated as a pair composed of a vector field and a tensor field, which is an extension of the classical interpretation of autonomous ODEs as vector fields. A method to obtain normal forms of such equations is also discussed and several results of calculations are given.

1. INTRODUCTION

The object of this paper is to study the ordinary differential equations (Abbrev. ODEs) of the following type including a small parameter ε:

$$\begin{cases} \varepsilon\dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} \qquad x \in \mathbf{R}^n,\ y \in \mathbf{R}^m, \quad (\dot{} = \tfrac{d}{dt}) \tag{1}$$

A typical example is the Van der Pol equation:

$$\begin{cases} \varepsilon\dot{x} = (x-x^3/3)+y \\ \dot{y} = -x. \end{cases} \tag{2}$$